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Variational inequalities for a body in a viscous shearing flow

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The slow motion of a body in a viscous shearing field is examined. Variational principles are used to derive inequalities which approximate the elements of the shearing matrix $\mathbf{M}$ of a body of arbitrary shape, where $\mathbf{M}$ is the matrix relating the force, torque and stresslet exerted by the body on the fluid to the relative translational and rotational velocities of the body and the rate of deformation of the undisturbed linear field. An upper bound for the elements of $\mathbf{M}$ is obtained by showing that the quadratic form of $\mathbf{M}$ increases monotonically with B, the region occupied by the body, while a lower bound for this form is given in terms of the electrostatic properties of a conductor and a dielectric of the same shape as B. Particular attention is paid to bodies of revolution, for which certain more definitive results are obtained: for example, their resistance to a rotation with axial symmetry is always less than twice their resistance to a rotation perpendicular to their axis.

1. Introduction

Exact analytic solutions to the Stokes equations, which describe creeping motions of bodies in a viscous fluid, are known for very few body shapes. This situation results from the difficulty of satisfying the boundary conditions in cases where the boundary does not form a co-ordinate surface of one of the few orthogonal co-ordinate systems for which the equations are separable. Recently, Youngren & Acrivos (1975) developed a numerical technique for solving these equations based on the fact that the problem of determining the slow viscous flow of an unbounded fluid past a solid particle can be formulated exactly as a system of integral equations of the first kind for a distribution of Stokeslets over the particle surface. In general, the quantities of primary interest are the force, the torque and the force dipole exerted by the body on the fluid, which, under creeping-flow conditions, are linearly related to the parameters of the undisturbed field. The coefficients in this relation comprise the elements of a *shearing matrix* **M** (Brenner & O'Neill 1972; Hinch 1972), which provides, in addition, the contribution of the body to the bulk stress tensor (see Batchelor 1970 for its definition) and thereby the rate of viscous dissipation arising from the presence of the particle. Knowledge of the shearing matrix also allows one to compute the motion imposed on a freely suspended body by a linear shear flow.

Although, in principle, **M** can be found numerically using the scheme proposed by Youngren & Acrivos (1975), such an approach becomes cumbersome for bodies of irregular shape. It is worthwhile, therefore, to develop methods for obtaining *ab initio* estimates for the elements of **M**, as well as methods for predicting how these are altered by a change in the geometry of the particle. In this way, for example, it is possible to determine **M** numerically for a class of bodies having simple shapes, and then to use the results below to arrive at approximate values for its elements corresponding to other geometries.

It is our aim here to present variational techniques for estimating the elements of the shearing matrix. The analysis will deal with problems involving bodies of arbitrary shape, with special attention given to bodies of revolution.

Maximum and minimum principles of entropy production and energy dissipation in Stokes flow and similar problems have been discussed by Helmholtz (1868), Korteweg (1883), Hill & Power (1956), Kearsley (1960), Keller, Rubenfeld & Molyneux (1967) and Weinberger (1972). Many studies have been concerned with obtaining bounds on the mass and momentum transport coefficients of suspensions (Prager 1963; Hashin 1969; Keller *et al.* 1967).

In order to identify the interesting parameters, we discuss in §2 the hydrodynamic problem for a body in an ambient linear field, the structure of the shearing matrix and its relation to the free-suspension problem. In §3 we derive two basic bounds for the shearing matrix. The first stems from a monotonicity property which is proved for the quadratic form of **M**, and the second is obtained in terms of the electrostatic capacity and polarization tensor associated with the body. The inequalities of §3 are applied in §§4 and 5 to various special linear flow fields which give rise to various principal minors of the shearing matrix. We obtain thereby an isoperimetric inequality for the average rotation resistance of the body and various inequalities for the eigenvalues of the tensors comprising the diagonal of **M**. Particular attention is paid to bodies of revolution. Results for some examples involving two tangential spheres of different sizes are computed.

2. The motion of a body in a viscous fluid

Consider a rigid body B with boundary $\dot{B}$ moving in an infinite domain. The body moves with a translational velocity U'_i and rotates about a fixed origin O with angular velocity Ω'_i. The domain D, which is the complement of B, is filled with viscous incompressible fluid which has velocity

$$u_{0i} = U_{0i} + e_{ij}x_j + \epsilon_{ijk}\omega_j x_k \qquad (2.1)$$

at infinity. Here x_i is the position vector with origin at O, U_{0i} is a constant translation, e_{ij} is the constant rate-of-strain tensor and $2\omega_i$ is the constant vorticity of the undisturbed flow. The summation convention is used throughout the paper and the Stokes approximation is assumed to be valid. Since we are considering only incompressible flows, e_{ij} must satisfy the condition $e_{kk} = 0$. The flow field in D is formulated as

$$u'_i = u_{0i} + u_i, \tag{2.2}$$

where u_i is the disturbance velocity field caused by the presence of the body. Here u_i and the pressure disturbance p satisfy the Stokes equations

$$\left.\begin{aligned} \sigma_{ij} &= \mu(u_{i,j}+u_{j,i}) - p\delta_{ij}, \\ \sigma_{ij,j} &= 0, \quad u_{i,i} = 0 \end{aligned}\right\} \tag{2.3}$$

and the boundary conditions

$$\left.\begin{aligned} u_i &= U_i + \epsilon_{ijk}\Omega_j x_k - e_{ij}x_j \quad \text{on} \quad \dot{B}, \\ u_i, p &\to 0 \quad \text{at} \quad \infty, \end{aligned}\right\} \tag{2.4}$$

where

$$U_i \equiv U'_i - U_{0i}, \quad \Omega_i \equiv \Omega'_i - \omega_i.$$

The force F_i, the torque T_i and the stresslet S_{ij} (see Batchelor 1970) exerted by the body on the fluid are defined by

$$F_i = \oint_{\dot{B}} \sigma_{ij} n_j \, dS, \tag{2.5}$$

$$T_i = \oint_{\dot{B}} \epsilon_{ijk} \sigma_{kl} x_j n_l \, dS, \tag{2.6}$$

$$S_{ij} = -\oint_{\dot{B}} [\tfrac{1}{2}(\sigma_{ik}x_j + \sigma_{jk}x_i - \tfrac{2}{3}\sigma_{kl}x_l\delta_{ij}) n_k - \mu(u_i n_j + u_j n_i)] \, dS, \tag{2.7}$$

where n_i is the inward normal on $\dot{B}$. Far from B the disturbances in the pressure, velocity and stress have asymptotic forms (Batchelor 1970) whose leading terms are directly related to the integral quantities F_i, T_i and S_{ij}:

$$4\pi p = F_j \frac{x_j}{r^3} - 3S_{jk}\frac{x_j x_k}{r^5} + \ldots, \tag{2.8}$$

$$4\pi\mu u_i = \tfrac{1}{2}F_j\left(\frac{\delta_{ij}}{r} + \frac{x_i x_j}{r^3}\right) + (S_{jk} + \epsilon_{jlk}T_l)\left(\frac{\delta_{ij}x_k - \delta_{ik}x_j}{r^3} - \frac{3}{2}\frac{x_i x_j x_k}{r^5}\right) + \ldots, \tag{2.9}$$

$$\begin{aligned} 4\pi\sigma_{ij} = F_k\left(-\frac{\delta_{ik}x_j + \delta_{jk}x_i}{2r^3} - 3\frac{x_i x_j x_k}{r^5}\right) &+ 3(S_{kl} + \epsilon_{kml}T_m) \\ &\times\left(-\frac{(\delta_{il}x_j + \delta_{jl}x_i)x_k}{r^5} + 5\frac{x_i x_j x_l x_k}{r^7}\right) + \ldots. \end{aligned} \tag{2.10}$$

The linearity of (2.3) and (2.4) implies that F_i, T_i and S_{ij} are related to U_i, Ω_i and e_{ij} by a linear relation of the form

$$\begin{pmatrix} F_i \\ T_i \\ S_{ij} \end{pmatrix} = \mu \begin{pmatrix} A_{ik} & D'_{ik} & Q'_{ikl} \\ D''_{ik} & B_{ik} & R'_{ikl} \\ Q''_{ijk} & R''_{ijk} & C_{ijkl} \end{pmatrix} \begin{pmatrix} U_k \\ \Omega_k \\ e_{kl} \end{pmatrix}, \tag{2.11}$$

in which the nine material tensors A_{ik}, etc., comprise the *shearing matrix* $\mathbf{M}$ for the body. If we define the 15-component vectors

$$\mathscr{F} = \{F_i, T_i, S_{ij}\}, \quad \mathscr{U} = \{U_i, \Omega_i, e_{ij}\}$$

the relation (2.11) becomes

$$\mathscr{F} = \mu \mathbf{M} \mathscr{U}.$$

It was shown by Hinch (1972) that the matrix $\mathbf{M}$ is symmetric and positive definite as long as one deals with e_{ij} which are symmetric and for which $e_{ii} = 0$. In order to define $\mathbf{M}$ uniquely for all e_{ij} and to keep it positive semidefinite and symmetric, we define $\mathbf{M}\mathscr{U}$ to be zero when $U_i = \Omega_i = 0$ and e_{ij} is either skew symmetric or equal to δ_{ij}. We further require that the S_{ij} in $\mathscr{F}$ be always symmetric and have $S_{ii} = 0$.

These normalizations mean that

$$Q'_{ikk} = R'_{ikk} = C_{ijkk} = 0,$$

and that

$$A_{ik} = A_{ki}, \quad B_{ik} = B_{ki},$$
$$C_{ijkl} = C_{jikl} = C_{ijlk} = C_{klij}, \quad D''_{ik} = D'_{ki},$$
$$Q''_{ijk} = Q''_{jik} = Q'_{kij}, \quad R''_{ijk} = R''_{jik} = R'_{kij}.$$

If one knows the quadratic form $\mathscr{U}.\mathbf{M}\mathscr{U}$ for traceless symmetric e_{ij}, one obtains $\mathscr{U}.\mathbf{M}\mathscr{U}$ for a general e_{ij} with the above normalization by replacing e_{ij} by

$$\tfrac{1}{2}(e_{ij}+e_{ji}) - \tfrac{1}{3}e_{kk}\delta_{ij}.$$

The quantities U_i, T_i and S_{ij} in (2.11) depend, of course, on the particular choice of the origin O.

Of special interest are bodies of revolution. If the origin is chosen to lie on the axis of symmetry of such a body and if p_i is a unit vector in the direction of the axis of symmetry, the elements of the shearing matrix take the simple forms

$$\left.\begin{aligned} A_{ij} &= a_1\delta_{ij} + (a_3 - a_1)p_ip_j, \\ B_{ij} &= b_1\delta_{ij} + (b_3 - b_1)p_ip_j, \\ C_{ijkl} &= c_1(p_ip_jp_kp_l - \tfrac{1}{3}p_ip_j\delta_{kl} - \tfrac{1}{3}\delta_{ij}p_kp_l + \tfrac{1}{9}\delta_{ij}\delta_{kl}) + \tfrac{1}{2}c_2(p_ip_jp_kp_l \\ &\quad - \delta_{il}p_jp_k - \delta_{ik}p_jp_l - \delta_{jk}p_ip_l - \delta_{jl}p_ip_k + \delta_{ij}p_kp_l + \delta_{kl}p_ip_j - \delta_{ij}\delta_{kl} \\ &\quad + \delta_{il}\delta_{jk} + \delta_{ik}\delta_{jl}) + c_3(-p_ip_jp_kp_l + \tfrac{1}{4}\delta_{il}p_jp_k + \tfrac{1}{4}\delta_{ik}p_jp_l + \tfrac{1}{4}\delta_{jk}p_ip_l \\ &\quad + \tfrac{1}{4}\delta_{jl}p_ip_k), \\ D'_{ij} &= D''_{ji} = d_1\epsilon_{ijk}p_k, \\ Q'_{kij} &= Q''_{ijk} = -q_1(\delta_{ik}p_j + \delta_{jk}p_i - 2p_ip_jp_k) - q_3(p_ip_jp_k - \tfrac{1}{3}\delta_{ij}p_k), \\ R'_{kij} &= R''_{ijk} = -r_1(\epsilon_{ikl}p_lp_j + \epsilon_{jkl}p_lp_i), \end{aligned}\right\} \quad (2.12)$$

where the scalar coefficients a_i, b_i, c_i, d_i, q_i and r_i depend only on the geometry of B and the choice of the origin O.

A problem of great interest is that of a freely suspended body. For such a body the velocity U_i and angular velocity Ω_i produced by a linear flow at infinity are determined from the fact that the force F_i and the torque T_i are zero. The vectors U_i and Ω_i are therefore found by solving the first six equations in (2.11) with $F_i = T_i = 0$. We thus obtain the velocity and angular velocity

$$\left.\begin{aligned} U_i &= -(E_{ij}Q'_{jkl} + G'_{ij}R'_{jkl})\,e_{kl}, \\ \Omega_i &= -(G''_{ij}Q_{jkl} + F_{ij}R'_{jkl})\,e_{kl}, \end{aligned}\right\} \quad (2.13)$$

where the matrix
$$\begin{pmatrix} E_{ij} & G'_{ij} \\ G''_{ij} & F_{ij} \end{pmatrix}$$
is the inverse of
$$\begin{pmatrix} A_{ij} & D'_{ij} \\ D''_{ij} & B_{ij} \end{pmatrix}.$$

Thus if the shearing matrix $\mathbf{M}$ is known, the coefficients in the relation (2.13) and hence the motion of a freely suspended body in a shearing field can be determined.

If B is axially symmetric about an axis which passes through the origin in the direction of the unit vector p_i, so that $\mathbf{M}$ has the form (2.12), then (2.13) becomes

$$\left.\begin{aligned} U_i &= \{\gamma(\tfrac{1}{2}\delta_{ik}p_l + \tfrac{1}{2}\delta_{il}p_k - p_i p_k p_l) + \beta(p_i p_k p_l - \tfrac{1}{3}p_i\delta_{kl})\}\, e_{kl}, \\ \Omega_i &= \tfrac{1}{2}b(\epsilon_{ijk}p_j p_l + \epsilon_{ijl}p_j p_k)\, e_{kl}, \end{aligned}\right\} \tag{2.14}$$

where
$$\gamma = 2\frac{d_1 dr_1 - b_1 q_1}{a_1 b_1 - d_1^2}, \quad \beta = \frac{q_3}{a_3}, \quad b = 2\frac{a_1 r_1 - d_1 q_1}{a_1 b_1 - d_1^2}. \qquad (2.15)\text{–}(2.17)$$

The constants γ, β and b have direct physical interpretations. If one takes the x_3 axis as the axis of symmetry, so that $p_i = \delta_{i3}$, then the axially symmetric shear $e_{33} = 1$, $e_{11} = e_{22} = -\frac{1}{2}$ produces axial translation

$$U_i = \beta\delta_{i3}, \quad \Omega_i = 0.$$

On the other hand, the two-dimensional shear $e_{ij} = \delta_{i1}\delta_{j3} + \delta_{i3}\delta_{j1}$ produces the motion

$$U_i = \gamma\delta_{i1}, \quad \Omega_i = -b\delta_{i2},$$

which represents an instantaneous rotation about the axis $x_1 = 0$, $x_3 = \gamma/b$ with angular velocity b.

We recall that, if the co-ordinates are fixed in space, the vector p_i along the axis of symmetry satisfies the equation

$$dp_i/dt = \epsilon_{ijk}(\Omega_j + \omega_j)\, p_k = b(e_{ik}p_k - e_{kl}p_k p_l p_i) + \epsilon_{ijk}\omega_j p_k,$$

which is the same as that of an ellipsoid of revolution of aspect ratio

$$\{(1-b)/(1+b)\}^{\frac{1}{2}}, \quad \text{provided that} \quad |b| \leqslant 1.$$

The motion of such an ellipsoid in a simple shear has been found by Jeffery (1922). Bretherton (1962) found the motion of a general body of revolution in an arbitrary uniform shear flow. In particular, he showed that there are bodies for which $|b| > 1$, and that the behaviour of such a body in a simple shear is quite different from that of an ellipsoid.

In general the body B also undergoes a translation relative to the fluid. However, when B is symmetric about the plane $x_3 = 0$, the constants β and γ are zero, so that no relative translation occurs at the origin.

We note that the C_{ijkl} do not enter the coefficients of (2.13). However, as we shall see, our approximations for the elements of Q'_{ijk} and R'_{ijk} will require a knowledge of the elements of C_{ijkl}.

3. Some general bounds

We begin by deriving the Helmholtz minimum principle for the problem (2.3) with (2.4).† We define the positive semidefinite symmetric bilinear functional

$$E(\mathbf{v},\mathbf{w}) \equiv \tfrac{1}{2}\mu\int_D (v_{i,j}+v_{j,i})(w_{i,j}+w_{j,i})\,dx$$

for smooth vector fields $\mathbf{v}$ and $\mathbf{w}$ in D which vanish sufficiently rapidly at infinity.

Consider any smooth vector field $\mathbf{v}$ which satisfies the conditions

$$\left.\begin{aligned} v_{i,i} = 0 \quad &\text{in} \quad D,\\ v_i = U_i + \epsilon_{ijk}\Omega_j x_k - e_{ik}x_k \quad &\text{on} \quad \dot{B},\\ v_i \to 0 \quad &\text{at} \quad \infty. \end{aligned}\right\} \tag{3.1}$$

If u_i is the solution of the problem (2.3) with (2.4), the condition $v_{i,i} = 0$ together with an integration by parts shows that

$$E(\mathbf{v},\mathbf{u}) = \int_D v_{i,j}\sigma_{ij}\,dx = \oint_{\dot{B}} v_i\sigma_{ij}n_j\,dS. \tag{3.2}$$

Since $v_i = u_i$ on $\dot{B}$, we see that

$$E(\mathbf{v},\mathbf{u}) = E(\mathbf{u},\mathbf{u}).$$

Then by Schwarz's inequality

$$E(\mathbf{v},\mathbf{u})^2 \leqslant E(\mathbf{v},\mathbf{v})\,E(\mathbf{u},\mathbf{u})$$

or

$$E(\mathbf{u},\mathbf{u}) \leqslant E(\mathbf{v},\mathbf{v}). \tag{3.3}$$

We see from (3.2) that $E(\mathbf{u},\mathbf{u})$ is the work done on the fluid by the boundary if the flow velocity is $\mathbf{u}$. Thus $E(\mathbf{u},\mathbf{u})$ is the rate of dissipation of energy for the flow field $\mathbf{u}$. The inequality (3.3) states that *among all motions which satisfy the boundary conditions and the incompressibility condition, Stokes flow minimizes the dissipation functional* $E(\mathbf{v},\mathbf{v})$. This is the Helmholtz (1868) principle.

Next, if we substitute the boundary values $v_i = u_i = U_i + (\epsilon_{ijk}\Omega_j - e_{ik})x_k$ in (3.2) and recall the definitions (2.5)–(2.7), we see that

$$E(\mathbf{u},\mathbf{u}) = U_iF_i + \Omega_iT_i + e_{ij}S_{ij} - 2\mu|B|e_{ij}e_{ij},$$

where $|B|$ denotes the volume of B. Hence, when $e_{ij} = e_{ji}$ and $e_{ii} = 0$, we have

$$\mathscr{U}.\mathbf{M}\mathscr{U} = \mu^{-1}E(\mathbf{u},\mathbf{u}) + 2|B|e_{ij}e_{ij}. \tag{3.4}$$

We shall use this identity and the Helmholtz principle to prove the following monotonicity theorem.

THEOREM 1.

Let B^* and B be two bodies with boundaries $\dot{B}^*$ and $\dot{B}$ and complementary domains D^* and D respectively. Let the solutions to the boundary-value prob-

† These are derived for smooth fields and boundaries. The extension to fields and boundaries which are not smooth is discussed by Weinberger (1972), whose notation we shall follow.

lem (2.3) with (2.4) with the same U_i, Ω_i and e_{ij} be u_i^* and u_i respectively. If B^* contains B, then the corresponding quadratic forms satisfy the inequality

$$\mathscr{U}.\mathbf{M}\mathscr{U} \leqslant \mathscr{U}.\mathbf{M}^*\mathscr{U}, \tag{3.5}$$

so that the difference matrix $\mathbf{M}^* - \mathbf{M}$ is always positive semidefinite. (Starred quantities refer to B^*.)

Proof. Because of our normalization of $\mathbf{M}$, we need to prove the inequality only when $e_{ij} = e_{ji}$ and $e_{ii} = 0$.

Let v_i be u_i^* in D^* and $U_i + \epsilon_{ijk}\Omega_j x_k - e_{ij}x_j$ in $D - D^*$. By (3.3) and (3.4)

$$\begin{aligned}\mathscr{U}.\mathbf{M}\mathscr{U} &= \mu^{-1}E(\mathbf{u},\mathbf{u}) + 2|B|e_{ij}e_{ij} \leqslant \mu^{-1}E(\mathbf{v},\mathbf{v}) + 2|B|e_{ij}e_{ij} \\ &= \mathscr{U}.\mathbf{M}^*\mathscr{U} + 2(|B| - |B^*|)\,e_{ij}e_{ij} + \frac{1}{2}\int_{D-D^*}(v_{i,j}+v_{j,i})(v_{i,j}+v_{j,i})\,dx \\ &= \mathscr{U}.\mathbf{M}^*\mathscr{U}.\end{aligned}$$

We shall discuss particular corollaries of this theorem in the following sections.

We have used the Helmholtz principle to show that the quadratic form of the shearing matrix $\mathbf{M}$ is a non-decreasing functional of B. This quadratic form can therefore be estimated from that of the shearing matrix of another body contained in B (or containing B), which provides a lower (or upper) bound. In many cases, however, this procedure may prove ineffective. To obtain a lower estimate for such cases, we shall make use of various functionals from the theory of electrostatic potentials. Since these functionals depend only on the solution of Laplace's equation, they are easier to estimate than the elements of $\mathbf{M}$.

To this end, consider the scalar and vector potential fields ϕ and ψ_i which are the solutions of the boundary-value problems

$$\left.\begin{array}{c}\phi_{,ii} = 0 \quad \text{in} \quad D, \\ \phi = 1 \quad \text{on} \quad \dot{B}, \quad \phi \to 0 \quad \text{as} \quad r \to \infty\end{array}\right\} \tag{3.6}$$

and

$$\left.\begin{array}{c}\psi_{i,jj} = 0 \quad \text{in} \quad D, \\ \psi_i = x_i \quad \text{on} \quad \dot{B}, \quad \psi_i \to 0 \quad \text{as} \quad r \to \infty.\end{array}\right\} \tag{3.7}$$

The electrostatic capacity C is defined by

$$C = \frac{1}{4\pi}\int_D \phi_{,i}\phi_{,i}\,dx, \tag{3.8}$$

and it follows from the divergence theorem that

$$\phi = C/r + O(r^{-2}) \quad \text{as} \quad r \to \infty.$$

Another important parameter is the centre g_i of the equilibrium charge distribution, which is defined (see Schiffer & Szegö 1949) by

$$g_i = \frac{1}{4\pi C}\oint_{\dot{B}} x_i \frac{\partial \phi}{\partial n}\,dS.$$

Integration by parts, the definitions of C and g_i, and the boundary conditions $\phi = 1$ and $\psi_i = x_i$ show that

$$\oint_{\dot{B}} \frac{\partial}{\partial n}(\psi_i - g_i\phi)\,dS = \oint_{\dot{B}} \phi\frac{\partial}{\partial n}(\psi_i - g_i\phi)\,dS = \oint_{\dot{B}} (\psi_i - g_i\phi)\frac{\partial \phi}{\partial n}\,dS = 0.$$

That is, the harmonic function $\psi_i - g_i\phi$ not only vanishes at infinity but has zero flux. Therefore

$$\psi_i = g_i\phi + O(r^{-2}) \quad \text{as} \quad r \to \infty. \tag{3.9}$$

A third useful quantity is the polarization tensor P_{ij}, which is defined by

$$P_{ij} = \int_D (\psi_{i,k} - g_i\phi_{,k})(\psi_{j,k} - g_j\phi_{,k})\,dx. \tag{3.10}$$

We now observe that, since u_i is solenoidal,

$$\int_D u_{i,j}u_{j,i}\,dx = \oint_{\dot{B}} u_i u_{j,i} n_j\,dS = \oint_{\dot{B}} u_i(u_{j,i}n_j - u_{j,j}n_i)\,dS.$$

Since each $n_j\partial/\partial x_i - n_i\partial/\partial x_j$ represents a tangential derivative on the boundary, we can evaluate the integral on the right by substituting the boundary values (2.4). In this way we find that

$$\begin{aligned} E(\mathbf{u},\mathbf{u}) &= \mu\int_D u_{i,j}u_{i,j}\,dx + \mu\int_D u_{i,j}u_{j,i}\,dx \\ &= \mu\int_D u_{i,j}u_{i,j}\,dx + 2\mu|B|\,\Omega_i\Omega_i - \mu|B|e_{ij}e_{ij}. \end{aligned}$$

Hence, we see from (3.4) that

$$\mathscr{U}.\mathbf{M}\mathscr{U} = \int_D u_{i,j}u_{i,j}\,dx + 2|B|\,\Omega_i\Omega_i + |B|e_{ij}e_{ij}. \tag{3.11}$$

However, since $\qquad u_i = U_i\phi + \epsilon_{ijk}\Omega_j\psi_k - e_{ij}\psi_j \quad \text{on} \quad \dot{B},$ (3.12)

and since both ϕ and ψ_i vanish at infinity, Dirichlet's principle shows that

$$\int_D u_{i,k}u_{i,k}\,dx \geqslant \int_D (U_i\phi_{,l} + \epsilon_{ijk}\Omega_j\psi_{k,l} - e_{ij}\psi_{j,l})(U_i\phi_{,l} + \epsilon_{ijk}\Omega_j\psi_{k,l} - e_{ij}\psi_{j,l})\,dx. \tag{3.13}$$

In view of (3.11) this inequality leads to a lower bound for the quadratic form $\mathscr{U}.\mathbf{M}\mathscr{U}$ when $e_{ij} = e_{ji}$ and $e_{ii} = 0$. In order to obtain a bound which is valid for arbitrary e_{ij} we need only replace e_{ij} by $\frac{1}{2}(e_{ij}+e_{ji}) - \frac{1}{3}e_{kk}\delta_{ij}$. We thus obtain the inequality

$$\begin{aligned} \mathscr{U}.\mathbf{M}\mathscr{U} \geqslant\; & 4\pi C\delta_{ij}U_iU_j + [4\pi C(g_kg_k\delta_{ij} - g_ig_j) + (P_{kk} + 2|B|)\,\delta_{ij} - P_{ij}]\,\Omega_i\Omega_j \\ & + [\tfrac{1}{4}(4\pi Cg_jg_l + P_{jl} + |B|\delta_{jl})\,\delta_{ik} + \tfrac{1}{4}(4\pi Cg_ig_l + P_{il} + |B|\delta_{il})\,\delta_{jk} \\ & + \tfrac{1}{4}(4\pi Cg_jg_k + P_{jk} + |B|\delta_{jk})\,\delta_{il} + \tfrac{1}{4}(4\pi Cg_ig_k + P_{ik} + |B|\delta_{ik})\,\delta_{jl} \\ & - \tfrac{1}{3}(4\pi Cg_ig_j + P_{ij} + |B|\delta_{ij})\,\delta_{kl} - \tfrac{1}{3}(4\pi Cg_kg_l + P_{kl} + |B|\delta_{kl})\,\delta_{ij} \\ & + \tfrac{1}{9}(4\pi Cg_mg_m + P_{mm} + 3|B|)\,\delta_{ij}\delta_{kl}]\,e_{ij}e_{kl} + 8\pi C\epsilon_{ijk}g_kU_i\Omega_j \\ & - 4\pi C(g_l\delta_{ik} + g_k\delta_{il} - \tfrac{2}{3}g_i\delta_{kl})\,U_ie_{kl} \\ & + [\epsilon_{ikm}(4\pi Cg_mg_l + P_{ml}) + \epsilon_{ilm}(4\pi Cg_mg_k + P_{mk})]\,\Omega_ie_{kl}. \end{aligned} \tag{3.14}$$

It is of some interest to know whether or not equality may hold in the lower bound (3.14). A partial answer is provided by the following theorem.

THEOREM 2.

Except in the trivial case $U_i = \Omega_i = \frac{1}{2}(e_{ij}+e_{ji}) - \frac{1}{3}e_{kk}\delta_{ij} = 0$, equality in (3.14) can only hold if the motion is perpendicular to some constant vector c_i at all points in D, leaves the point g_i fixed and satisfies the inequality

$$\tfrac{1}{4}(e_{ij}+e_{ji})(e_{ij}+e_{ji}) - \tfrac{1}{3}e_{ii}e_{jj} < 2\Omega_i\Omega_i. \tag{3.15}$$

Moreover, if $e_{ij} = 0$, equality holds if and only if B is axially symmetric and the boundary motion $U_i + \epsilon_{ijk}\Omega_j x_k$ represents a rotation about the axis of symmetry.

Proof. We assume without loss of generality that $e_{ij} = e_{ji}$ and $e_{kk} = 0$.

By Dirichlet's principle, equality in (3.13) holds if and only if each component u_i of the solution of (2.3) with (2.4) is harmonic; that is, if and only if

$$u_i = U_i\phi + \epsilon_{ijk}\Omega_j\psi_k - e_{ik}\psi_k. \tag{3.16}$$

The function on the right satisfies all the conditions of (2.3) and (2.4) with $p = 0$ except possibly the divergence condition. Thus, equality holds in (3.14) if and only if the right-hand side of (3.16) is solenoidal; that is,

$$U_i\phi_{,i} + \epsilon_{ijk}\Omega_j\psi_{k,i} - e_{ik}\psi_{k,i} = 0. \tag{3.17}$$

Expanding in spherical harmonics, we see from (3.9) that there exist constants α_{ij} such that

$$\psi_i = g_i\phi + \alpha_{ij}x_j r^{-3} + O(r^{-3}).$$

Hence the condition (3.17) can be written as

$$\{U_i + (\epsilon_{ijk}\Omega_j - e_{ik})g_k\}\phi_{,i} + (\epsilon_{ijk}\Omega_j - e_{ik})\alpha_{kl}(x_l/r^3)_{,i} + O(r^{-4}) = 0.$$

Since

$$\phi_{,i} = -C(x_i/r^3) + O(r^{-3}),$$

we see that we must have

$$U_i + (\epsilon_{ijk}\Omega_j - e_{ik})g_k = 0. \tag{3.18}$$

That is, the point g_i is at rest under the prescribed motion. Moreover, equating the terms of order r^{-3} to zero, we find that

$$\{(\epsilon_{mjk}\Omega_j - e_{mk})\alpha_{km}\delta_{il} - 3(\epsilon_{ijk}\Omega_j - e_{ik})\alpha_{kl}\}x_i x_l = 0$$

for all x_i. This implies that the coefficient matrix is skew symmetric, so that

$$(\epsilon_{ijk}\Omega_j - e_{ik})\alpha_{kl} + (\epsilon_{ljk}\Omega_j - e_{lk})\alpha_{ki} = \lambda\delta_{il} \tag{3.19}$$

for some λ.

An application of the divergence theorem to (3.10) together with the spherical-harmonic expansion of ψ_i shows that

$$\begin{aligned} P_{ij} &= \lim_{R\to\infty}\oint_{r=R}(\psi_i - g_i\phi)\frac{\partial}{\partial n}(\psi_j - g_j\phi)\,dS + \oint_{\dot{B}}(\psi_i - g_i\phi)\frac{\partial}{\partial n}(\psi_j - g_j\phi)\,dS \\ &= \oint_{\dot{B}}(x_i - g_i)\frac{\partial}{\partial n}(\psi_j - g_j\phi)\,dS \\ &= \oint_{\dot{B}}(\psi_j - g_j\phi)\frac{\partial}{\partial n}(x_i - g_i)\,dS \\ &\qquad - \lim_{R\to\infty}\oint_{r=R}\left[(x_i - g_i)\frac{\partial}{\partial n}(\psi_j - g_j\phi) - (\psi_j - g_j\phi)\frac{\partial}{\partial n}(x_i - g_i)\right]dS \\ &= -|B|\delta_{ij} + 4\pi\alpha_{ij}, \end{aligned}$$

so that
$$\alpha_{ij} = (4\pi)^{-1}\{P_{ij} + |B|\delta_{ij}\}.$$
Therefore the matrix α_{ij} is symmetric and positive semidefinite.

We rotate co-ordinates so that α_{ij} is diagonal. If one of the diagonal elements is zero we see from (3.19) that $\lambda = 0$. If the diagonal elements of α_{ij} are all positive and $\lambda \neq 0$, we see from (3.19) that the diagonal elements of the matrix $\epsilon_{ijk}\Omega_j - e_{ik}$ all have the same sign. Since the trace of this matrix is zero, this is impossible. Thus the constant λ in (3.19) must be zero.

It can be shown (Weinberger 1973) that at most one of the diagonal elements of α_{ij} is zero. It follows that, in the co-ordinate system where α_{ij} is diagonal, the diagonal elements of e_{ij} are zero, while
$$e_{12} = \frac{\alpha_{11}-\alpha_{22}}{\alpha_{11}+\alpha_{22}}\Omega_3$$
with similar expressions for e_{13} and e_{23}. The inequality (3.15) follows immediately.

It also follows from the form of e_{ij} when α_{ij} is diagonal that the matrix $\epsilon_{ijk}\Omega_j - e_{ik}$ must have the form
$$\epsilon_{ijk}\Omega_j - e_{ik} = \alpha_{ip}\epsilon_{pqk}m_q,$$
where the vector m_q is the solution of the system
$$(\alpha_{kk}\delta_{ij} - \alpha_{ij})\,m_j = 2\Omega_i.$$
But since the matrix $\epsilon_{qpk}m_q$ is skew symmetric, its determinant is zero. Therefore, there exists a non-zero constant vector c_i such that
$$c_i(\epsilon_{ijk}\Omega_j - e_{ik}) = 0.$$
Since, by (3.16) and (3.18),
$$u_i = (\epsilon_{ijk}\Omega_j - e_{ik})(\psi_k - \phi g_k),$$
we see that $c_i u_i = 0$. This proves the first statement of the theorem. We also note that the above expression for u_i implies that $u_i = O(r^{-2})$ as $r \to \infty$ on account of (3.9). Thus it follows that, for equality in (3.14), there must be no net force F_i on B.

Suppose now that $e_{ij} = 0$. If Ω_i is also zero, the fact that the velocity vanishes at g_i shows that $U_i = 0$. This is the trivial case. If $e_{ij} = 0$ but $\Omega_i \neq 0$, then clearly $c_i = \Omega_i$. By (2.4), (3.18) and (3.16) the quantity $(x_i - g_i)\,u_i = 0$ on $\dot{B}$ and at infinity. Since u_i is harmonic and solenoidal, $(x_i - g_i)\,u_i$ is also harmonic. Hence
$$(x_i - g_i)\,u_i = 0 \quad \text{in} \quad D.$$

We now choose new co-ordinates with the origin at g_i and with the x_3 axis in the direction of the vector $c_i = \Omega_i$. Then $g_i = 0$, $U_i = 0$, $\Omega_1 = \Omega_2 = 0$ and we must have
$$u_1 = -hx_2, \quad u_2 = hx_1, \quad u_3 = 0, \tag{3.20}$$
where h is a scalar function. The divergence condition shows that $x_2 h_{,1} - x_1 h_{,2} = 0$. If we introduce cylindrical co-ordinates (ρ, θ, z), this condition becomes $\partial h/\partial\theta = 0$, so that $h = h(\rho, z)$. The fact that the u_i are harmonic and the boundary conditions show that h is the solution of boundary-value problem
$$\left.\begin{aligned} h_{,\rho\rho} + (3/\rho)\,h_{,\rho} + h_{,zz} &= 0, \\ h = \Omega_3 \quad \text{on} \quad \dot{B}, \quad h \to 0 \quad &\text{at} \quad \infty. \end{aligned}\right\} \tag{3.21}$$

If $\Omega_3 \neq 0$, the maximum principle shows that $|h| < |\Omega_3|$ in D. If there were a point (ρ_0, θ_0, z_0) on $\dot{B}$ and a point (ρ_0, θ_1, z_0) in D, then we would have

$$h(\rho_0, z_0) = \Omega_3 \quad \text{and also} \quad |h(\rho_0, z_0)| < |\Omega_3|.$$

This contradiction shows that B must be axially symmetric about the x_3 axis.

Conversely, if B is axially symmetric about the x_3 axis and if h is the solution of the problem (3.21), then the vector field (3.20) is harmonic and solves the problem (2.3) and (2.4) with $p = 0$ and $e_{ij} = U_i = \Omega_1 = \Omega_2 = 0$. Thus the theorem is proved.

We shall discuss the various submatrices of **M** and (3.14) in the following sections.

4. The diagonal tensors

If we choose $\Omega_i = e_{ij} = 0$, the inequality (3.5) becomes

$$A_{ij} U_i U_j \leqslant A^*_{ij} U_i U_j. \tag{4.1}$$

It follows that all the diagonal elements, principal minors and eigenvalues of the matrix $A^*_{ij} - A_{ij}$ are non-negative. For example, if B and B^* have axial symmetry about the x_3 axis, so that both A_{ij} and A^*_{ij} are of the form (2.12), we find that

$$a_1 \leqslant a^*_1, \quad a_3 \leqslant a^*_3. \tag{4.2}$$

Moreover, we find from (3.14) and theorem 2 that, if $U_i U_i > 0$, then

$$(A_{ij} - 4\pi C\delta_{ij}) U_i U_j > 0, \tag{4.3}$$

which becomes

$$a_1 > 4\pi C, \quad a_3 > 4\pi C \tag{4.4}$$

when B is axially symmetric. Both (4.1) and (4.3) were recently obtained by Weinberger (1972) in terms of the settling speed. Weinberger also indicated that (4.4) is sharp in the sense that for a spheroid with a symmetry axis of length l and perpendicular axis of length l/r_e the ratio $a_1/4\pi C$ approaches unity as $r_e \to \infty$ (r_e is the aspect ratio).

Consider next the case in which

$$U_i = 0, \quad e_{ij} = 0 \tag{4.5}$$

for all i and j. Inequality (3.5) then reduces to the inequality

$$B_{ij} \Omega_i \Omega_j \leqslant B^*_{ij} \Omega_i \Omega_j \tag{4.6}$$

for the rotation tensor. Thus, as with the translation tensor, we find that the diagonal elements, principal minors and eigenvalues of $B^*_{ij} - B_{ij}$ are non-negative. In particular for bodies of revolution we obtain

$$b^*_1 \geqslant b_1, \quad b^*_3 \geqslant b_3, \tag{4.7}$$

where b_1 and b_3 are defined in (2.12). Also, (3.14) and theorem 2 yield the inequality

$$[B_{ij} - 4\pi C(g_k g_k \delta_{ij} - g_i g_j) - (P_{kk} + 2|B|)\delta_{ij} + P_{ij}]\,\Omega_i \Omega_j \geqslant 0, \tag{4.8}$$

with equality for $\Omega_i \neq 0$ if and only if B is axially symmetric, the origin is on the axis of symmetry and Ω_i is parallel to this axis.

If B is axially symmetric and the origin is on the axis of symmetry, then

$$g_i = gp_i \tag{4.9}$$

and the polarization tensor is of the form

$$P_{ij} = \pi_1 \delta_{ij} + (\pi_3 - \pi_1) p_i p_j, \tag{4.10}$$

where p_i is a unit vector along the axis of revolution. We then obtain the estimates

$$b_3 = 2(\pi_1 + |B|) \tag{4.11}$$

and

$$b_1 \geqslant 4\pi C g^2 + \pi_1 + \pi_3 + 2|B|, \tag{4.12}$$

with equality if and only if B is a sphere centred at the origin.

It follows from (4.8) that the trace of the coefficient matrix is non-negative, so that

$$B_{ii} \geqslant 8\pi C g_i g_i + 2P_{ii} + 6|B|, \tag{4.13}$$

with equality if and only if B is a sphere centred at the origin. The isoperimetric inequality

$$C \geqslant (3|B|/4\pi)^{\frac{1}{3}}$$

has been obtained by Poincaré, Faber and Szegö (see Polya & Szegö 1951). The isoperimetric inequality

$$P_{ii} \geqslant 6|B|$$

was established by Schiffer (1957). Substituting these inequalities in (4.13), we find that

$$\tfrac{1}{3}B_{ii} \geqslant 6|B| + (\tfrac{8}{3}\pi)^{\frac{2}{3}} (2|B|)^{\frac{1}{3}} g_i g_i \geqslant 6|B|. \tag{4.14}$$

Equality is obtained when B is a sphere and the origin is at its centre. We may therefore state (4.14) as an isoperimetric inequality: among all bodies of given volume the value of the average rotational resistance attains its minimum only when the body is a sphere centred at the origin.

Next, if we divide the inequality (4.12) by (4.11), we obtain the lower bound

$$\frac{b_1}{b_3} \geqslant \tfrac{1}{2} + \frac{\pi_3 + |B| + 4\pi C g^2}{2(\pi_1 + |B|)}, \tag{4.15}$$

with equality only for a sphere centred at the origin. It follows that

$$b_1/b_3 > \tfrac{1}{2}$$

for all axially symmetric bodies. It is not known what the greatest lower bound of the ratio b_1/b_3 is, but $b_1/b_3 = 1$ for a flat disk (see for example Happel & Brenner 1965, p. 173). The minimum value of this ratio among oblate spheroids is 0·7961, which is attained when the ratio of the principal axes is 0·408.

The third interesting tensor in the diagonal of the shearing matrix $\mathbf{M}$ is the tensor C_{ijkl}. It describes the stresslet imposed by a stationary body on a fluid experiencing a linear deformation at infinity, and the resulting rate of dissipation. When $U_i = \Omega_i = 0$ the inequality (3.5) becomes

$$e_{ij} C^*_{ijkl} e_{kl} \geqslant e_{ij} C_{ijkl} e_{kl}. \tag{4.16}$$

Since C_{ijkl} may be thought of as a matrix whose indices are the pairs ij and kl, the inequality (4.16) shows that this matrix has the same monotonicity properties as A_{ij} and B_{ij}.

We now consider a body B which is axially symmetric about the x_3 axis, so that (2.12) holds with $p_i = \delta_{i3}$. We then see from (2.12) that for those e_{ij} for which $e_{ij} = e_{ji}$ and $e_{ii} = 0$ we have

$$C_{ijkl} e_{ij} e_{kl} = c_1 e_{33}^2 + \tfrac{1}{2} c_2 [(e_{11} - e_{22})^2 + 4e_{12}^2] + c_3 (e_{13}^2 + e_{23}^2). \tag{4.17}$$

Particular ambient flows provide inequalities for the constants appearing in (4.17). For example, the flow with

$$e_{ij} = \delta_{i1}\delta_{j2} + \delta_{i2}\delta_{j1}$$

yields

$$c_2^* \geqslant c_2. \tag{4.18}$$

Similarly, for an ambient field in which $e_{ij} = \delta_{i1}\delta_{j3} + \delta_{i3}\delta_{j1}$ we obtain

$$c_3^* \geqslant c_3. \tag{4.19}$$

Finally, by considering the axisymmetric extensional flow $e_{ij} = \delta_{ij} - 3\delta_{i3}\delta_{j3}$ at infinity, we find that

$$c_1^* \geqslant c_1. \tag{4.20}$$

A lower bound for C_{ijkl} for a body of arbitrary shape involving the body's capacity and polarization tensor is obtained from (3.14) by setting $U_i = \Omega_i = 0$. We find that the symmetric matrix $C_{ijkl} - \hat{C}_{ijkl}$, where

$$\begin{aligned}
\hat{C}_{ijkl} = {} & \tfrac{1}{4}(4\pi C g_j g + P_{jl} + |B|\delta_{jl})\,\delta_{ik} \\
& + \tfrac{1}{4}(4\pi C g_i g_l + P_{il} + |B|\,\delta_{il})\,\delta_{jk} \\
& + \tfrac{1}{4}(4\pi C g_j g_k + P_{jk} + |B|\delta_{jk})\,\delta_{il} \\
& + \tfrac{1}{4}(4\pi C g_i g_k + P_{ik} + |B|\delta_{ik})\,\delta_{jl} \\
& - \tfrac{1}{3}(4\pi C g_i g_j + P_{ij} + |B|\,\delta_{ij})\,\delta_{kl} \\
& - \tfrac{1}{3}(4\pi C g_k g_l + P_{kl} + |B|\delta_{kl})\,\delta_{ij} \\
& + \tfrac{1}{9}(4\pi C g_m g_m + P_{mm} + 3|B|)\,\delta_{ij}\delta_{kl},
\end{aligned}$$

is positive definite. For an axially symmetric body we then find that, in terms of the components of g_i and P_{ij}, defined in (4.9) and (4.10),

$$c_2 > \pi_1 + |B|, \tag{4.21}$$

$$c_3 > 4\pi C g^2 + \pi_1 + \pi_3 + 2|B|, \tag{4.22}$$

$$c_1 > 4\pi C g^2 + \tfrac{1}{2}(\pi_1 + 2\pi_3 + 3|B|). \tag{4.23}$$

Quantitative comparisons between these estimates and some exact results are shown in table 1 and figure 1. Table 1 gives the ratios of c_2, c_3 and c_1 to their corresponding estimated values from (4.21)–(4.23) for three types of spheroids: a thin oblate spheroid ($r_e \ll 1$), a sphere ($r_e = 1$) and a slender prolate spheroid ($r_e \gg 1$) (r_e is the ratio of the length of the axis of revolution to the length of the axis perpendicular to it). The c's were calculated from Jeffery's (1922) solution. The second example, shown in figure 1, compares the calculated values of c_2, c_3

	Thin oblate spheroid $r_e \ll 1$	Sphere $r_e = 1$	Slender prolate spheroid $r_e \gg 1$
$\dfrac{c_2}{\frac{1}{2}(P_{kk} - P_{ij}p_ip_j + 2\|B\|)}$	$\frac{4}{3} + O(r_e)$	$\frac{5}{3}$	$\dfrac{2r_e^2}{\log 2r_e} + O\left(\dfrac{1}{r_e^2}\right)$ $\left(\text{Note that } c_2 \to \dfrac{4}{3r_e^2}\right)$
$\dfrac{c_3}{4\pi Cg^2 + \frac{1}{2}(P_{kk} + P_{ij}p_ip_j + 4\|B\|)}$	$2 + O(r_e)$	$\frac{5}{3}$	$2 + O\left(\dfrac{1}{r_e^2 \log 2r_e}\right)$
$\dfrac{c_1 + \frac{1}{2}c_2}{4\pi Cg^2 + \frac{1}{4}(P_{kk} + 3P_{ij}p_ip_j + 6\|B\|)}$	$2 + O(r_e)$	$\frac{5}{3}$	$1 + O\left(\dfrac{1}{r_e^2 \log 2r_e}\right)$

TABLE 1. The ratio of c_2, c_3 and c_1 to their corresponding estimates from (4.21), (4.22) and (4.23) for spheroids. r_e is the ratio of the length of the axis of revolution to the length of the axis perpendicular to it. c_1, c_3 and g are calculated relative to the centroid.

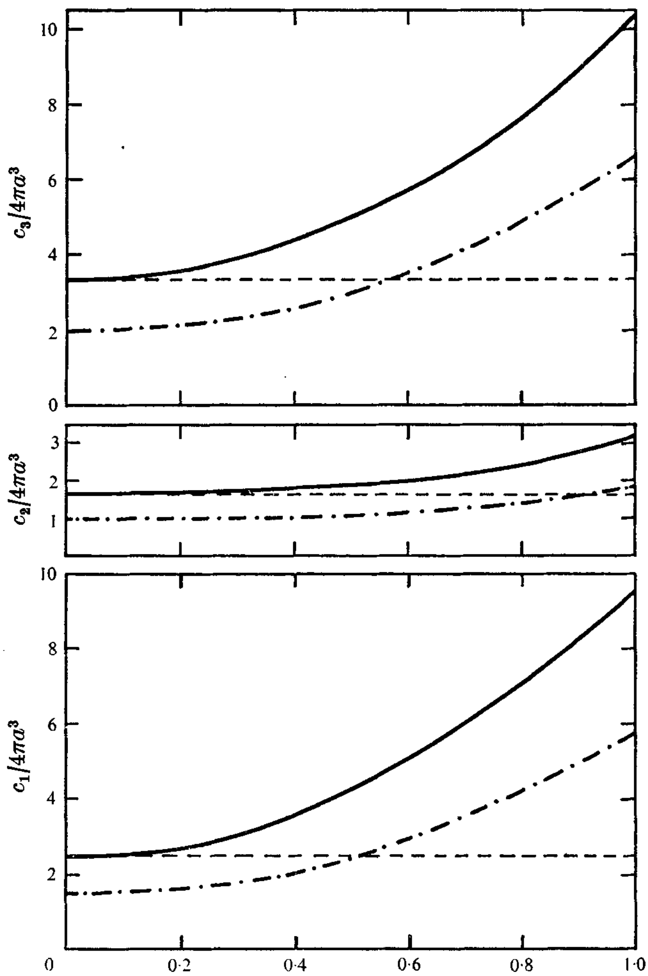

FIGURE 1. Comparison of calculated values of c_2, c_3 and c_1 for pair of tangential spheres (full lines) with lower bounds obtained from, respectively, (4.21), (4.22) and (4.23) (dotted lines) and from, respectively, (4.18), (4.19) and (4.20) (dashed lines) using the larger sphere of radius a as the largest contained body.

and c_1 for pairs of tangential spheres (Nir & Acrivos 1973) with lower bounds obtained on the basis of (4.21)–(4.23) and by means of the monotonicity property stated in (4.18)–(4.20), with the larger sphere taken as the largest contained body. The polarization tensors for both table 1 and figure 1 are quoted from Schiffer & Szegö (1949) and the centre of the equilibrium charge distribution on the surface of tangential spheres is stated elsewhere (Nir 1973).

It can be seen that the worst estimates from (4.22) and (4.23) are obtained for the sphere. However, (4.23) is sharp in the sense that for a prolate spheroid the ratio

$$c_1/[\tfrac{1}{2}(\pi_1+2\pi_3+3|B|)+4\pi Cg^2]$$

approaches unity as $r_e \to \infty$. Furthermore, it is evident from figure 1 that, as would be expected in view of the results in table 1, the monotonicity inequality leads to better bounds for spheres and nearly spherical bodies. For tangential spheres of almost equal size, however, the monotonicity bounds become inferior to those obtained from the polarization tensor.

5. The coupling tensors and the free-suspension coefficients

The fact that the determinants of the principal minors of a positive semi-definite matrix are non-negative allows us to obtain, from theorem 1 and (3.14), inequalities involving the off-diagonal matrices D_{ij}, Q_{ijk} and R_{ijk}. Such inequalities will, however, also involve at least two diagonal elements.

For example, if B is axially symmetric about the x_3 axis, we find from theorem 2 that

$$(d_1-4\pi Cg)^2 < [a_1-4\pi C][b_1-4\pi Cg^2-\pi_1-\pi_3-2|B|]. \tag{5.1}$$

(Here a_1, b_1 and d_1 are defined in (2.12) and g, π_1 and π_3 in (4.9) and (4.10).) This inequality is better than the well-known inequality $d_1^2 < a_1 b_1$ (see for example Happel & Brenner 1965, p. 178), which follows from the positive definiteness of $\mathbf{M}$. The inequality (5.1) gives useful bounds for d_1 if one has good upper bounds for the diagonal elements a_1 and b_1. By similar reasoning we see from theorem 1 that, if B^* contains B, then

$$(d_1^*-d_1)^2 \leqslant (a_1^*-a_1)(b_1^*-b_1). \tag{5.2}$$

Similar inequalities hold for the elements of Q'_{ijk} and R'_{ijk}. Thus, if B is axially symmetric about the x_3 axis and we take the principal minor of the rows and columns which correspond to U_3 and e_{33}, we find the inequalities

$$(q_3+4\pi Cg)^2 \leqslant \tfrac{4}{9}(a_3-4\pi C)(c_1-\pi_3-\tfrac{1}{2}\pi_1-\tfrac{3}{2}|B|-4\pi Cg^2) \tag{5.3}$$

and

$$(q_3^*-q_3)^2 \leqslant (a_3^*-a_3)(c_1^*-c_1) \tag{5.4}$$

when B^* contains B. Since the constant β in (2.14) is equal to q_3/a_3, bounds for q_3 and a_3 give bounds for β.

Similarly, if we consider the minor which comes from the rows and columns corresponding to U_1 and e_{13}, we find that

$$(q_1-4\pi Cg)^2 \leqslant \tfrac{1}{4}(a_1-4\pi C)(c_3-\pi_1-\pi_3-2|B|-4\pi Cg^2) \tag{5.5}$$

and

$$(q_1^*-q_1)^2 \leqslant \tfrac{1}{4}(a_1^*-a_1)(c_3^*-c_3) \tag{5.6}$$

when B^* contains B, while the principal minor corresponding to Ω_2 and e_{13} yields

$$(r_1-\pi_3+\pi_1+4\pi Cg^2)^2 \leqslant \tfrac{1}{4}(b_1-\pi_1-\pi_3-2|B|-4\pi Cg^2)(c_3-\pi_1-\pi_3-2|B|-4\pi Cg^2) \quad (5.7)$$

and $$(r_1^*-r_1)^2 \leqslant \tfrac{1}{4}(b_1^*-b_1)(c_1^*-c_3) \quad (5.8)$$

when B^* contains B.

The inequalities (5.1), (5.2) and (5.5)–(5.8) can be used to approximate d_1, q_1 and r_1. Since the methods of §4 serve to approximate a_1 and b_1, we can in principle approximate all the quantities that occur in the definitions (2.15) and (2.17) of γ and b. However, since such inequalities also involve at least two diagonal elements, their use in approximating the free-suspension coefficients β, γ and b through (2.15)–(2.17) may prove inefficient.

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